Output feedback stabilization for a scalar conservation law with a nonlocal velocity

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Abstract

In this paper, we study the output feedback stabilization for a scalar conservation law with a nonlocal velocity, that models a highly re-entrant manufacturing system as encountered in semi-conductor production. By spectral analysis, we obtain a complete result on the exponential stabilization for the linearized control system. Moreover, by using a Lyapunov function approach, we also prove the exponential stabilization results for the nonlinear control system in certain cases.

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1 Introduction

In this paper, we study the scalar conservation law

$$\rho_t(t,x) + (\rho(t,x)\lambda(W(t)))_x = 0, \quad t \in (0,+\infty), x \in (0,1), \tag{1.1}$$

where

$$W(t) = \int_0^1 \rho(t,x)dx. \tag{1.2}$$

We assume that the velocity function λ is in $C^1(\mathbb{R};(0,+\infty))$. Let us recall that the special case

$$\lambda(s) = \frac{1}{1+s}, \quad s \in [0,+\infty),$$

was, for example, used in [6, 26].

In the manufacture system, the initial data is given as

$$\rho(0,x) = \rho_0(x), \quad x \in (0,1). \tag{1.3}$$

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For this control system, the control is the influx

$$u(t) := \rho(t,0)\lambda(W(t)), \tag{1.4}$$

and the measurement is the outflux

$$y(t) := \rho(t,1)\lambda(W(t)). \tag{1.5}$$

We consider the following output feedback law

$$u(t) - \overline{\rho}\lambda(\overline{\rho}) = k(y(t) - \overline{\rho}\lambda(\overline{\rho})), \quad t \in (0, +\infty), \tag{1.6}$$

in which $k \in \mathbb{R}$ is a tuning parameter and $\overline{\rho} \in \mathbb{R}$ is the equilibrium that we want to stabilize when time t goes to $+\infty$.

The conservation law that we study here is used to model the semiconductor manufacturing systems, see e.g. [6, 24, 26]. These systems are characterized by their highly re-entrant feature with very high volume (number of parts manufactured per unit time) and very large number of consecutive production steps as well. The main character of this partial differential equation model is described in terms of the velocity function λ which is a function of the total mass $W(t)$ (the integral of the density ρ).

The control problems for conservation laws and general hyperbolic equations/systems have been widely studied. The controllability of nonlinear hyperbolic equations (or systems) are studied in [10, 14, 22, 28, 29, 36] for solutions without shocks, and in [1, 2, 3, 4, 7, 19, 25, 32] for solutions with shocks. As for asymptotic stability/stabilization of hyperbolic equations (systems), two main strategies have been used. The first one relies on a careful analysis of the evolution of the solution along the characteristic curves; see in particular [5, 7, 20, 27, 31, 33]. The second one relies mainly on a Lyapunov function approach; see, in particular, [11, 12, 13, 17, 18, 21, 35, 37].

Concerning the manufacturing model of (1.1) itself, an optimal control problem, motivated by [6, 26], related to the *Demand Tracking Problem* was studied in [15] (see also [34] for a generalized system where $\lambda = \lambda(x, W(t))$). The objective of that optimal control problem is to minimize, by choosing influx $u(t) = h(t)$ instead of (1.6), the L^p-norm ($p \geq 1$) of the difference between the actual out-flux $y(t) := \rho(t,1)\lambda(W(t))$ and a given demand forecast $y_d(t)$ over a fixed time period. This is an open-loop control system. Another related work [8], which is also motivated in part by [6, 26], addressed well-posedness for systems of hyperbolic conservation laws with a nonlocal velocity in $\mathbb{R}^n$. The authors studied the Cauchy problem in the whole space $\mathbb{R}^n$ without considering any boundary conditions and they gave a necessary condition for the possible optimal controls. In a recent paper [16], controllability of solution and out-flux for (1.1) have been obtained by the same authors of this one.

In this paper, we study the exponential stability of the manufacturing system (1.1) under the feedback law (1.6). The problem of *exponential stabilization* can be described as follows:

For any given equilibrium $\overline{\rho} \in \mathbb{R}$ and any initial data ρ_0, does there exist $k \in \mathbb{R}$ such that $\overline{\rho}$ is exponentially stable for the closed-loop control system (1.1), (1.3) together with (1.6), namely, the weak solution ρ to the Cauchy problem (1.1), (1.3) and (1.6) converges to $\overline{\rho}$ exponentially when time t goes to $+\infty$?

If $\overline{\rho} = 0$, the situation is simple. Natural feedback laws can drive the state to zero exponentially fast. In particular, the zero control produces a solution which vanishes after a finite time. Nevertheless, if $\overline{\rho} \neq 0$, the situation is much more complicated. More precisely, the stabilization results depend on the equilibrium $\overline{\rho} \neq 0$ and the velocity function λ through the following quantity

$$d := \frac{\overline{\rho}\lambda'(\overline{\rho})}{\lambda(\overline{\rho})}. \tag{1.7}$$

First we establish a complete result for the linearized system by spectral approach. A sufficient and necessary condition of exponential stability of the linearized system is given in Theorem 3.1. However, an example (see [23, page 285]) shows that an arbitrary small perturbation of the characteristic speeds of a linear hyperbolic system may break the stability property, hence it seems difficult to deduce the exponential stability of the original nonlinear problem from the exponential stability of the linearized system. In order to overcome this difficulty, we also use Lyapunov function approach to prove exponential stability for the linearized system and, then, use the same Lyapunov function to prove the (local) exponential stability for the nonlinear system. The Lyapunov functions that we construct in this paper are inspired by [9, 13, 35, 37]. However they have to be modified according to the nonlocal feature of the nonlinear system.

The structure of this paper is as follows: In Section 2, we prove the well-posedness of the nonlocal closed-loop system. Then, in Section 3, we study the exponential stability of the linearized system. The main results on the stabilization of the nonlinear problem, Theorem 4.1 and Theorem 4.2, and their proofs, are given in Section 4.

2 Preliminaries

In order to stabilize the system by feedback controls, we first need to recall the usual definition of a weak solution to the Cauchy problem (see, e.g., [11, Section 2.1]) and then prove that the closed-loop system is well-posed.

Definition 2.1. Let $\overline{\rho} \in \mathbb{R}$, $p \in [1, +\infty)$, $k \in \mathbb{R}$ and $\rho_0 \in L^1(0,1)$ be given. A weak solution of the Cauchy problem

$$\begin{cases} \rho_t(t,x) + (\rho(t,x)\lambda(W(t)))_x = 0, \quad t \in (0,+\infty), x \in (0,1), \\ \rho(0,x) = \rho_0(x), \quad x \in (0,1), \\ u(t) - \overline{\rho}\lambda(\overline{\rho}) = k(y(t) - \overline{\rho}\lambda(\overline{\rho})), \quad t \in (0,+\infty) \end{cases} \tag{2.1}$$

is a function $\rho \in C^0([0,+\infty); L^p(0,1))$ such that, for every $T > 0$, every $\tau \in [0,T]$ and every $\varphi \in C^1([0,\tau] \times [0,1])$ such that

$$\varphi(\tau,x) = 0, \forall x \in [0,1] \quad \text{and} \quad \varphi(t,1) = 0, \forall t \in [0,\tau],$$

one has

$$-\int_0^\tau \int_0^1 \rho(t,x)(\varphi_t(t,x) + \lambda(W(t))\varphi_x(t,x))dxdt - \int_0^1 \rho_0(x)\varphi(0,x)dx \\ + \int_0^\tau \Big(y(t)\varphi(t,1) - [ky(t) + (1-k)\overline{\rho}\lambda(\overline{\rho})]\varphi(t,0)\Big)dt = 0.$$

Then, the following Lemma holds.

Lemma 2.1. *Let $\overline{\rho} \in \mathbb{R}$, $p \in [1,+\infty)$ and $k \in \mathbb{R}$ be given. For any given $\rho_0 \in L^p(0,1)$, the Cauchy problem (2.1) has a unique weak solution $\rho \in C^0([0,+\infty); L^p(0,1))$. Moreover, for every $T > 0$, the following maps*

$$\rho_0 \in L^p(0,1) \mapsto \rho \in C^0([0,T]; L^p(0,1)), \tag{2.2}$$

$$\rho_0 \in L^p(0,1) \mapsto (u,y) \in L^p(0,T) \times L^p(0,T), \tag{2.3}$$

are continuous.

Proof. The proof of Lemma 2.1 is similar to that of Theorem 2.3 in [15], thus we only show the main ideas and omit the details of the proof.

Introduce the characteristic curve:

$$\frac{d\xi}{ds} = \lambda(W(s)), \quad s \geq 0,$$

where $W(s) = \int_0^1 \rho(s,x)dx$. Since ρ is constant along the characteristics, one can define a solution candidate in terms of ξ as following

$$\rho(t,x) = \begin{cases} \rho_0(x - \xi(t)), & \text{if } 0 \leq \xi(t) \leq x \leq 1, \\ k\rho(\xi^{-1}(\xi(t)-x),1) + \dfrac{(1-k)\overline{\rho}\lambda(\overline{\rho})}{\lambda(W(\xi^{-1}(\xi(t)-x)))}, & \\ \qquad \text{if } 0 \leq x \leq \xi(t) - n + 1 \leq 1, \text{ or } 0 \leq \xi(t) - n \leq x \leq 1 \text{ for } n \in \mathbb{N}. & \end{cases} \tag{2.4}$$

Then by contraction mapping principle and fixed point argument as in [15], one can prove that (2.4) is indeed the unique weak solution to the original system (2.1). Thanks to (2.4), one can get the continuity of the maps (2.2) and (2.3): See the proof of [34, Theorem 4.1].

□

Our next lemma is straightforward and we omit its proof.

Lemma 2.2. *Let $\overline{\rho} \in \mathbb{R}$ and $k \in \mathbb{R}$ be given. If $\rho_0 \in C^1([0,1])$ satisfies the C^1 compatibility conditions*

$$\begin{cases} \lambda(\int_0^1 \rho_0(x)dx)(\rho_0(0) - k\rho_0(1)) - (1-k)\overline{\rho}\lambda(\overline{\rho}) = 0, \\ \lambda(\int_0^1 \rho_0(x)dx)(\rho_0'(0) - k\rho_0'(1)) - \lambda'(\int_0^1 \rho_0(x)dx)(\rho_0(0) - \rho_0(1))(\rho_0(0) - k\rho_0(1)) = 0, \end{cases}$$

then the Cauchy problem (2.1) admits a unique classical solution $\rho \in C^1([0,+\infty) \times [0,1])$.

3 Stabilization to $\overline{\rho}$ for the linearized system

Before studying the nonlinear control system (2.1), we first linearize it near $\overline{\rho} \in \mathbb{R}$ and then study the linearized closed loop system:

$$\begin{cases} \widetilde{\rho}_t(t,x) + \lambda(\overline{\rho})\widetilde{\rho}_x(t,x) = 0, \quad t \in (0,+\infty), x \in (0,1), \\ \widetilde{\rho}(0,x) = \widetilde{\rho}_0(x), \quad x \in (0,1), \\ \widetilde{u}(t) = k\widetilde{y}(t) + (k-1)d\widetilde{W}(t), \quad t \in (0,+\infty), \end{cases} \tag{3.1}$$

where d is given by (1.7) and

$$\widetilde{W}(t) := \int_0^1 \widetilde{\rho}(t,x)dx, \quad \widetilde{u}(t) := \widetilde{\rho}(t,0)\lambda(\overline{\rho}), \quad \widetilde{y}(t) := \widetilde{\rho}(t,1)\lambda(\overline{\rho}).$$

The stability result for (3.1) can be stated as follows:

Theorem 3.1. *Let $\overline{\rho} \in \mathbb{R}$ be a constant. Then, $0 \in L^2(0,1)$ is exponentially stable in $L^2(0,1)$ for the closed loop system (3.1) if and only if $d > -1$ and $|k| < 1$. That is to say: if and only if $d > -1$ and $|k| < 1$, there exist constants $C = C(\overline{\rho}, k) > 0$ and $\alpha = \alpha(\overline{\rho}, k) > 0$ such that the following holds: For any $\rho_0 \in L^2(0,1)$, the weak solution $\rho \in C^0([0,+\infty); L^2(0,1))$ to the Cauchy problem (3.1) satisfies*

$$\|\widetilde{\rho}(t,\cdot)\|_{L^2(0,1)} \leq Ce^{-\alpha t}\|\widetilde{\rho}_0\|_{L^2(0,1)}, \quad \forall t \in [0,+\infty). \tag{3.2}$$

Next, we prove Theorem 3.1 by spectral analysis in Section 3.1. While in Section 3.2, we give another proof, relying on a Lyapunov function approach, that $d > -1$ and $|k| < 1$ imply that $0 \in L^2(0,1)$ is exponentially stable for the closed loop system (3.1). The Lyapunov functions, which are constructed in Section 3.2, will be used for the stabilization of the nonlinear control system as well.

3.1 Proof of Theorem 3.1 by spectral analysis

Without loss of generality, we assume

$$\lambda(\overline{\rho}) = 1.$$

Otherwise, a scaling transformation $t \mapsto \frac{t}{\lambda(\overline{\rho})}$ can easily make it. Thus, by (1.7),

$$d = \overline{\rho}\lambda'(\overline{\rho}).$$

Then we omit the $\widetilde{}$ symbols in Section 3.1 and Section 3.2, and rewrite (3.1) into the following

$$\begin{cases} \rho_t(t,x) + \rho_x(t,x) = 0, \quad t \in (0,+\infty), x \in (0,1), \\ \rho(0,x) = \rho_0(x), \quad x \in (0,1), \\ \rho(t,0) = k\rho(t,1) + (k-1)dW(t), \quad t \in (0,+\infty), \end{cases} \tag{3.3}$$

where $W(t) = \int_0^1 \rho(t,x)dx$. The exponential decay estimate (3.2) turns to

$$\|\rho(t,\cdot)\|_{L^2(0,1)} \leq Ce^{-\alpha t}\|\rho_0\|_{L^2(0,1)}, \quad \forall t \in [0,+\infty). \tag{3.4}$$

Applying the results in [30], we have the following propositions.

Proposition 3.1. *Let $k \in \mathbb{R}$ and $\rho_0 \in L^2(0,1)$ be given. The Cauchy problem (3.3) has a unique solution $\rho \in C^0([0,+\infty); L^2(0,1))$.*

Proposition 3.2. *Let $S(t)$ $(t \geq 0)$ be the C_0 semigroup on $L^2(0,1)$ that corresponds to the solution map of (3.3) and A be the infinitesimal generator of the semigroup $S(t)$ $(t \geq 0)$. Denote $\sigma_p(A)$ and $\sigma(A)$ as the point spectrum and the spectrum of A, respectively. Then,*

$$\sigma_p(A) = \sigma(A).$$

Proposition 3.3. *Let $\omega(A) := \inf\{\omega \in \mathbb{R} \mid \exists M = M(\omega) : \|S(t)\| \leq Me^{\omega t}, \forall t \geq 0\}$ and $s(A) := \sup\{\Re(\mu) \mid \mu \in \sigma(A)\}$, where $\Re(\mu)$ denotes the real part of μ. One has*

$$\omega(A) = s(A).$$

Obviously, by Proposition 3.2 and Proposition 3.3, Theorem 3.1 is equivalent to the following Lemma:

Lemma 3.1. *One has $s(A) < 0$ if and only of $d > -1$ and $|k| < 1$.*

Proof of Lemma 3.1: We only need to study the eigenvalues of the system (3.3). Let $\mu \in \mathbb{C}$ be an eigenvalue of the system (3.3) and $\phi \neq 0$ be a corresponding eigenfunction. The pair (μ, ϕ) satisfies

$$\begin{cases} \mu\phi(x) + \phi'(x) = 0, \quad x \in (0,1), \\ \phi(0) = k\phi(1) + (k-1)d \displaystyle\int_0^1 \phi(x)dx. \end{cases}$$

For $\mu \in \mathbb{C}$, the existence of $\phi \neq 0$ such that (3.1) holds is equivalent to

$$1 - ke^{-\mu} + (1-k)d\int_0^1 e^{\mu x}dx = 0, \tag{3.5}$$

the corresponding eigenfunction being $\phi(x) = e^{-\mu x}$ (up to a multiplicative factor).

Then we analyze the solution for the characteristic equation (3.5) in various cases.

Case 1. $d = -1$ and $k \in \mathbb{R}$.

Obviously, (3.5) admits a zero eigenvalue $\mu = 0$ if and only if

$$(1+d)(1-k) = 0. \tag{3.6}$$

Hence, if $d = -1$, (3.5) admits a solution $\mu = 0$ which shows immediately that $0 \in L^2(0,1)$ is not asymptotically stable for (3.1) whatever $k \in \mathbb{R}$ is.

Case 2. $d \neq -1$ and $k = 1$.

In view of (3.6), (3.5) admits a solution $\mu = 0$ and thus $0 \in L^2(0,1)$ is not asymptotically stable for (3.1).

Case 3. $d \neq -1$ *and* $k \neq 1$.

We need to analyze the nontrivial solution of the following equation:

$$1 - ke^{-\mu} + d(1-k)\frac{1-e^{-\mu}}{\mu} = 0, \quad \mu \neq 0.$$

It is equivalent to study the zero points of the following continuous function:

$$f_{d,k}(\mu) := \begin{cases} 1 - ke^{-\mu} + d(1-k)\dfrac{1-e^{-\mu}}{\mu}, & \text{if } \mu \neq 0, \\ (1+d)(1-k), & \text{if } \mu = 0. \end{cases} \tag{3.7}$$

Case 3.1. $d \neq -1$ *and* $|k| > 1$.

We will apply degree theory for homotopic functions (see [11, Appendix B]) to show that $f_{d,k}(\mu)$ has infinite zero points in the right half plane $\{\mu \in \mathbb{C} | \Re(\mu) > 0\}$, and, therefore, $0 \in L^2(0,L)$ is not stable for (3.1). In fact, $f_{d,k}$ behaves close to $1 - ke^{-\mu}$ as $|\mu| \to +\infty$.

Let

$$H(\theta, d, k, \mu) := f_{\theta d,k}(\mu) = 1 - ke^{-\mu} + \theta d(1-k)\frac{1-e^{-\mu}}{\mu}, \quad \mu \neq 0. \tag{3.8}$$

Then, in particular, $H(0,d,k,\mu) = f_{0,k}(\mu) = 1 - ke^{-\mu}$ and $H(1,d,k,\mu) = f_{d,k}(\mu)$. Obviously, $f_{0,k}$ vanishes at $\mu_{k,n}$ with

$$\mu_{k,n} := \begin{cases} \ln k + i2n\pi, & \text{if } \quad k > 0, \\ \ln|k| + i(2n+1)\pi, & \text{if } \quad k < 0, \end{cases} \qquad \forall n \in \mathbb{Z}.$$

Note that, since $|k| > 1$, $\Re\mu_{k,n} > 0$. For any fixed $n \in \mathbb{Z}$ and $\varepsilon > 0$, let

$$\Omega^{\varepsilon}_{k,n} := \{\mu \in \mathbb{C} || 1 - ke^{-\mu}| < \varepsilon \text{ and } |\mu - \mu_{k,n}| < 1\} \subset \mathbb{C}.$$

It is easy to see that $\Omega^{\varepsilon}_{k,n}$ is a bounded open set of $\mathbb{C}$ and

$$\deg(f_{0,k}(\mu), \Omega^{\varepsilon}_{k,n}, 0) = 1, \quad \forall n \in \mathbb{Z}. \tag{3.9}$$

One also easily checks that, if $\varepsilon > 0$ is small enough,

$$\partial\Omega^{\varepsilon}_{k,n} \subset \{\mu \in \mathbb{C} || 1 - ke^{-\mu}| = \varepsilon\}, \quad \forall n \in \mathbb{Z}, \tag{3.10}$$

$$\Omega^{\varepsilon}_{k,n} \subset \{\mu \in \mathbb{C} | \Re(\mu) > 0\}, \quad \forall n \in \mathbb{Z}. \tag{3.11}$$

We now fix $\varepsilon > 0$ small enough so that (3.10) and (3.11) hold. Notice that, for every $\theta \in [0,1]$ and any $\mu \in \partial\Omega^{\varepsilon}_{k,n}$,

$$|H(\theta,d,k,\mu)| = |1 - ke^{-\mu} + \theta d(1-k)\frac{1-e^{-\mu}}{\mu}| \geq \varepsilon - |\theta(1-k)||d|\frac{1 + \dfrac{1+\varepsilon}{|k|}}{|\mu_{k,n}| - 1}.$$

Hence, for any fixed $\varepsilon > 0$, there exists $N \in \mathbb{Z}^+$ such that: for all $n \in \mathbb{Z}, |n| > N$,

$$|H(\theta, d, k, \mu)| \geq \frac{\varepsilon}{2} > 0, \quad \forall \theta \in [0,1], \forall \mu \in \partial\Omega^{\varepsilon}_{k,n}.$$

Applying degree theory [11, Appendix B], we get for all $n \in \mathbb{Z}, |n| > N$ and all $\theta \in [0,1]$ that

$$\begin{aligned}\deg(f_{d,k}(\mu), \Omega^{\varepsilon}_{k,n}, 0) &= \deg(H(1,d,k,\mu), \Omega^{\varepsilon}_{k,n}, 0) = \deg(H(\theta,d,k,\mu), \Omega^{\varepsilon}_{k,n}, 0)\\ &= \deg(H(0,d,k,\mu), \Omega^{\varepsilon}_{k,n}, 0) = \deg(f_{0,k}(\mu), \Omega^{\varepsilon}_{k,n}, 0) = 1.\end{aligned}$$

Therefore, $f_{d,k}$ has one zero point in $\Omega^{\varepsilon}_{k,n} \subset \{\mu \in \mathbb{C} | \Re(\mu) > 0\}$ for every $n \in \mathbb{Z}, |n| > N$.

Case 3.2. $d < -1$ and $-1 \leq k < 1$.

Notice that $f_{d,k}(0) = (1+d)(1-k) < 0$ and that $f_{d,k}(\mu) \to 1$ as $\mu \to +\infty$ with $\mu \in \mathbb{R}$. These facts together with the continuity imply that $f_{d,k}$ has at least one zero point in $(0, +\infty)$.

Case 3.3. $d > -1$ and $k = -1$.

We prove that in this case $f_{d,-1}$ has infinite zero points on the imaginary axis. Let $\mu = ib$ ($b \in \mathbb{R} \setminus \{0\}$) be such that

$$f_{d,-1}(ib) = 1 + e^{-ib} + 2d\frac{1 - e^{-ib}}{ib} = 0, \tag{3.12}$$

i.e.,

$$ib(1 + \cos b - i \sin b) + 2d(1 - \cos b + i \sin b) = 0.$$

Hence $f_{d,-1}(ib) = 0$ if and only if

$$b(1 + \cos b) + 2d \sin b = 0, \tag{3.13}$$

$$b \sin b + 2d(1 - \cos b) = 0. \tag{3.14}$$

Note that (3.13) and (3.14) together with $b \neq 0$ imply that $\cos b \neq 1$. Then, using also the identity

$$b(1 + \cos b)(1 - \cos b) = b \sin^2 b,$$

one gets that $b \in \mathbb{R} \setminus \{0\}$ is a solution of (3.12) if and only if $b \in \mathbb{R} \setminus \{0\}$ satisfies

$$g(b) := \frac{b \sin b}{2(\cos b - 1)} - d = -\frac{b}{2} \cot \frac{b}{2} - d = 0. \tag{3.15}$$

Obviously, for any fixed $d > -1$, $g(\cdot) \in C^0(2n\pi, 2(n+1)\pi)$ for all $n \in \mathbb{Z}$ and

$$\begin{aligned} g(b) &\to -\infty, \quad \text{as} \quad b \to 2n\pi^-, \forall n \in \mathbb{Z}^+,\\ g(b) &\to +\infty, \quad \text{as} \quad b \to 2(n+1)\pi^+, \forall n \in \mathbb{Z}^+. \end{aligned}$$

Therefore, $g(\cdot)$ vanishes at least once in each interval $(2n\pi, 2(n+1)\pi)$ for $n \in \mathbb{Z}^+$, which implies that $f_{d,-1}$ has infinite zero points on the imaginary axis.

Case 3.4. $d > -1$ and $|k| < 1$.

We apply degree theory [11, Appendix B] again for homotopic functions to show firstly that $s(A) \leq 0$, namely, $f_{d,k}$ has no zero points in the right half plane $\{\mu \in \mathbb{C} | \Re(\mu) \geq 0\}$.

Let $H(\theta, d, k, \mu)$ be defined by (3.8) and for $R > 0$,

$$\Omega_R := \{\mu \in \mathbb{C} | \Re(\mu) > 0 \text{ and } |\mu| < R\}.$$

For any $R > 0$, $H(0, d, k, \mu) = f_{0,k}(\mu)$ has no zero points in Ω_R since $|f_{0,k}(\mu)| \geq 1 - |ke^{-\mu}| \geq 1 - |k| > 0$ for all $\mu \in \mathbb{C}, \Re(\mu) \geq 0$.

Then we claim that: for $R > 0$ sufficiently large

$$H(\theta, d, k, \mu) = f_{\theta d,k}(\mu) \neq 0, \quad \forall \theta \in [0, 1], \forall \mu \in \{\mu \in \mathbb{C} | \Re(\mu) \geq 0 \text{ and } |\mu| \geq R\}, \tag{3.16}$$

Property (3.16) readily follows from

$$|f_{\theta d,k}(\mu)| \geq 1 - |k| - 2\frac{|\theta d||1-k|}{|\mu|}, \quad \forall \mu \in \mathbb{C} \setminus \{0\} \text{ such that } \Re(\mu) \geq 0.$$

Next we claim that: for $R > 0$ sufficiently large,

$$H(\theta, d, k, \mu) = f_{\theta d,k}(\mu) \neq 0, \quad \forall \mu \in \partial\Omega_R. \tag{3.17}$$

To prove (3.17), one first points out that, by (3.16), it is sufficient to prove that $f_{\theta d,k}$ does not vanish on the imaginary axis. We use a contradiction argument to prove this fact.

Let $\mu = ib$ ($b \in \mathbb{R} \setminus \{0\}$) be a zero point of $f_{\theta d,k}$, i.e.,

$$f_{\theta d,k}(ib) = 1 - ke^{-ib} + \theta d(1-k)\frac{1 - e^{-ib}}{ib} = 0. \tag{3.18}$$

Property (3.18) is equivalent to

$$b(1 - k\cos b) + \theta d(1-k)\sin b = 0, \tag{3.19}$$

$$-kb\sin b + \theta d(1-k)(1 - \cos b) = 0. \tag{3.20}$$

Multiplying (3.19) with $1 - \cos b$ and (3.20) with $\sin b$, one gets that

$$b(1 - k\cos b)(1 - \cos b) = -kb\sin^2 b, \tag{3.21}$$

Equality (3.21) is equivalent to

$$b(1+k)(1 - \cos b) = 0,$$

which implies $\cos b = 1$, and thus $\sin b = 0$, since $b \neq 0$ and $|k| < 1$. Substituting $\cos b = 1$ and $\sin b = 0$ into (3.19), we get that $k = 1$. It is a contradiction with the fact that $|k| < 1$. This concludes the proof of (3.17).

Property (3.17) and the degree theory for homotopic functions for $H(\theta, d, k, \mu)$ give that

$$\begin{aligned}\deg(f_{d,k}(\mu), \Omega_R, 0) &= \deg(H(1, d, k, \mu), \Omega_R, 0) \\ &= \deg(H(0, d, k, \mu), \Omega_R, 0) = \deg(f_{0,k}(\mu), \Omega_R, 0) = 0.\end{aligned}$$

Therefore, $f_{d,k}$ dose not vanish in Ω_R and further in the right half plane $\{\mu \in \mathbb{C} | \Re(\mu) \geq 0\}$, namely, $s(A) \leq 0$.

Finally, we show that $s(A) < 0$. For any fixed $d > -1$ and $k \in (-1,1)$, there exists $r > 0$ such that $1 - ke^{-\mu} \geq 1 - |k||e^{-\mu}| = 1 - |k|e^{-\Re(\mu)} > 0$ for all $\mu \in \{\mu \in \mathbb{C} | \Re(\mu) \geq -r\}$. Since $f_{d,k}(\mu) \approx 1 - ke^{-\mu}$ as $|\mu|$ tends to $+\infty$, there exists then $R > 0$ such that $f_{d,k}(\mu) \neq 0$ for all $\mu \in \{\mu \in \mathbb{C} | \Re(\mu) \geq -r, |\mu| > R\}$. If $s(A) < -r$, we are done. Otherwise, $-r \leq s(A) \leq 0$, then $s(A)$ must be achieved by some $\mu \in \{\mu \in \mathbb{C} | f_{d,k}(\mu) = 0, |\mu| \leq R, -r \leq \Re(\mu) \leq 0\}$ since $\mu \mapsto f_{d,k}(\mu)$ is continuous. Note that $f_{d,k}(\mu) = 0$ has no solution on the imaginary axis, we conclude that $s(A) < 0$ which concludes the proof of Lemma 3.1 and thus of Theorem 3.1. $\square$

3.2 Proof of Theorem 3.1 for the case $d > -1$ and $|k| < 1$ by a Lyapunov function approach

Here, using a Lyapunov function approach, we give another proof of the exponential stability if $d > -1$ and $|k| < 1$. Note that the solution map of Cauchy problem (3.3) defines a C_0 semigroup $S(t)$ $(t \geq 0)$ on $L^2(0,1)$, without loss of generality, it suffices to construct the Lyapunov function for every classical (i.e. C^1) solutions. An important fact is that, as we will see in Section 4, the same Lyapunov function also works for the nonlinear closed loop system.

We divide our proof into two cases: $|d| < 1$ and $d \geq 1$.

Case 1: $|d| < 1$.

We construct a Lyapunov function as follows:

$$L(t) := \int_0^1 e^{-\beta x} \rho^2(t,x)dx + aW^2(t), \quad \forall t \in [0,+\infty), \tag{3.22}$$

where the constants $\beta > 0$ and $a \in \mathbb{R}$ are chosen later. (The introduction of $e^{-\beta x}$ is motivated by [9, 13, 35, 37].) By the definition of $W(t)$ and the Cauchy-Schwarz inequality, we know

$$W^2(t) \leq \int_0^1 e^{\beta x} dx \int_0^1 e^{-\beta x} \rho^2(t,x)dx = \frac{e^\beta - 1}{\beta} \int_0^1 e^{-\beta x} \rho^2(t,x)dx. \tag{3.23}$$

If

$$a > -\frac{\beta}{e^\beta - 1}, \tag{3.24}$$

$L(t)$ is positive definite for all $t \geq 0$ and there exists two constants $C_i = C_i(a,\beta) > 0$ $(i = 1,2,3,4)$ such that

$$\begin{aligned} C_1 \|\rho(t,\cdot)\|^2_{L^2(0,1)} \leq C_2 &\int_0^1 e^{-\beta x} \rho^2(t,x)dx \leq L(t) \\ &\leq C_3 \int_0^1 e^{-\beta x} \rho^2(t,x)dx \leq C_4 \|\rho(t,\cdot)\|^2_{L^2(0,1)}, \quad \forall t \in [0,+\infty). \end{aligned} \tag{3.25}$$

Let us first compute the time derivative of L along the solution of (3.3). Note that

$$\dot{W}(t) = -\int_0^1 \rho_x(t,x)dx = \rho(t,0) - \rho(t,1). \tag{3.26}$$

It follows from (3.22), (3.26) and (3.3) that

$$
\begin{aligned}
\dot{L}(t) &= -\int_0^1 e^{-\beta x}(\rho^2(t,x))_x dx + 2aW(t)\dot{W}(t) \\
&= -\beta \int_0^1 e^{-\beta x}\rho^2(t,x)dx - \Big[e^{-\beta x}\rho^2(t,x)\Big]_{x=0}^{x=1} + 2aW(t)(\rho(t,0)-\rho(t,1)) \\
&= -\beta \int_0^1 e^{-\beta x}\rho^2(t,x)dx + (k^2 - e^{-\beta})\rho^2(t,1) \\
&\quad + 2(k-1)(a+kd)\rho(t,1)W(t) + d(k-1)\big(2a+(k-1)d\big)W^2(t).
\end{aligned}
$$

Let $\beta > 0$ be small enough so that

$$
e^{-\beta} > k^2. \tag{3.27}
$$

Then

$$
\begin{aligned}
\dot{L}(t) = &-\beta \int_0^1 e^{-\beta x}\rho^2(t,x)dx \\
&+ (k^2 - e^{-\beta})\left[\rho(t,1) - \frac{(k-1)(a+kd)}{e^{-\beta}-k^2}W(t)\right]^2 \\
&+ \left[d(k-1)\big(2a+(k-1)d\big) + \frac{(k-1)^2(a+kd)^2}{e^{-\beta}-k^2}\right]W^2(t) \\
\le &-\beta \int_0^1 e^{-\beta x}\rho^2(t,x)dx \\
&+ \frac{1-k}{e^{-\beta}-k^2}\left[(1-k)a^2 + 2ad(k-e^{-\beta}) + e^{-\beta}(1-k)d^2\right]W^2(t) \\
= &-\beta \int_0^1 e^{-\beta x}\rho^2(t,x)dx \\
&+ \frac{[(k-1)a + (e^{-\beta}-k)d]^2}{e^{-\beta}-k^2}W^2(t) + d^2(1-e^{-\beta})W^2(t).
\end{aligned}
$$

Take

$$
a := \frac{e^{-\beta}-k}{1-k}\,d, \tag{3.28}
$$

which verifies (3.24) since $|d| < 1$ and (3.27). Moreover, by (3.23) and (3.28), we get

$$
\dot{L}(t) \le -\beta\big[1 - d^2(e^{\beta}-1)^2 e^{-\beta}\beta^{-2}\big]\int_0^1 e^{-\beta x}\rho^2(t,x)dx. \tag{3.29}
$$

As $\beta \to 0+$, $1 - d^2(e^{\beta}-1)^2 e^{-\beta}\beta^{-2} \to 1 - d^2 > 0$. Hence, taking $\beta > 0$ small enough, we may assume that

$$
1 - d^2(e^{\beta}-1)^2 e^{-\beta}\beta^{-2} > 0 \tag{3.30}
$$

From (3.29) and (3.30), one has

$$
\dot{L}(t) \le -\frac{\beta}{C_3}\big[1 - d^2(e^{\beta}-1)^2 e^{-\beta}\beta^{-2}\big]L(t)
$$

Therefore, there exists a constant $\alpha = \alpha(\overline{\rho}, k) > 0$ such that

$$\dot{L}(t) \leq -\alpha L(t), \quad \forall t \in [0, +\infty). \tag{3.31}$$

Finally, we conclude from the fact that

$$L(0) \leq C_3 \int_0^1 e^{-\beta x} \rho_0^2(x) dx \leq C_3 \|\rho_0\|^2_{L^2(0,1)} \tag{3.32}$$

and (3.31) that there exists a constants $C = C(\overline{\rho}, k) > 0$ such that (3.4) holds. This concludes the proof of Theorem 3.1 for the case $|d| < 1$.

Case 2. $d \geq 1$.

In this case, the construction of Lyapunov function is not as direct as in the case $|d| < 1$. Let

$$V_1(t) := \int_0^1 \rho^2(t, x) dx + bW^2(t), \quad \forall t \in [0, +\infty),$$

where $b \in (0, +\infty)$ is a constant to be determined. We compute the time derivative of V_1 along the solution of (3.3). One has

$$\begin{aligned}
\dot{V}_1(t) :&= -\int_0^1 (\rho^2(t, x))_x dx + 2bW(t)\dot{W}(t) \\
&= \rho^2(t, 0) - \rho^2(t, 1) + 2bW(t)(\rho(t, 0) - \rho(t, 1)) \\
&= (k^2 - 1)\rho^2(t, 1) + 2(k - 1)(b + kd)\rho(t, 1)W(t) + d(k - 1)[2b + (k - 1)d]W^2(t).
\end{aligned}$$

Since $k \in (-1, 1)$, in order to then deduce that $\dot{V}_1(t) \leq 0$, it suffices to require that

$$4(k - 1)^2(b + kd)^2 - 4d(k^2 - 1)(k - 1)[2b + (k - 1)d] \leq 0,$$

that is,

$$4(k - 1)^2(b - d)^2 \leq 0.$$

Taking

$$b := d,$$

readily gives

$$V_1(t) = \int_0^1 \rho^2(t, x) dx + dW^2(t), \tag{3.33}$$

$$\dot{V}_1(t) = (k^2 - 1)(\rho(t, 1) + dW(t))^2. \tag{3.34}$$

Let

$$\xi(t, x) := \rho(t, x) + dW(t), \quad t \in (0, +\infty), x \in (0, 1). \tag{3.35}$$

By (3.34) and (3.35)

$$\dot{V}_1(t) = (k^2 - 1)\xi^2(t, 1). \tag{3.36}$$

From (3.3) and (3.35), one gets that ξ satisfies the following Cauchy problem

$$\begin{cases} \xi_t(t,x) + \xi_x(t,x) = d\dot{W}(t), \quad t \in (0,+\infty), x \in (0,1), \\ \xi(0,x) = \rho_0(x) - \overline{\rho} + dW(0), \quad x \in (0,1), \\ \xi(t,0) = k\xi(t,1), \quad t \in (0,+\infty). \end{cases} \tag{3.37}$$

Let

$$V_2(t) := \int_0^1 e^{-x}\xi^2(t,x)dx, \quad \forall t \in [0,+\infty). \tag{3.38}$$

Then

$$\begin{aligned} \dot{V}_2(t) &:= 2\int_0^1 e^{-x}\xi(t,x)\xi_t(t,x))dx \\ &= 2\int_0^1 e^{-x}\xi(t,x)(-\xi_x(t,x) + d\dot{W}(t))dx \\ &= -\Big[e^{-x}\xi^2(t,x)\Big]_{x=0}^{x=1} - \int_0^1 e^{-x}\xi^2(t,x)dx + 2d\dot{W}(t)\int_0^1 e^{-x}\xi(t,x)dx \\ &= -e^{-1}\xi^2(t,1) + \xi^2(t,0) - V_2(t) + 2d(\xi(t,0) - \xi(t,1))\int_0^1 e^{-x}\xi(t,x)dx \\ &= (k^2 - e^{-1})\xi^2(t,1) - V_2(t) + 2d(k-1)\xi(t,1)\int_0^1 e^{-x}\xi(t,x)dx. \end{aligned} \tag{3.39}$$

By the Cauchy-Schwarz inequality,

$$\Big|\int_0^1 e^{-x}\xi(t,x)dx\Big|^2 \leq (1-e^{-1})V_2(t). \tag{3.40}$$

From (3.39) and (3.40), there exists a constant $A = A(d,k) > 0$ such that

$$\dot{V}_2(t) \leq -\frac{1}{2}V_2(t) + A\xi^2(t,1). \tag{3.41}$$

Let

$$V(t) := \frac{2A}{1-k^2}V_1(t) + V_2(t), \quad \forall t \in [0,+\infty). \tag{3.42}$$

Combining (3.36), (3.41) and (3.42), one has

$$\dot{V}(t) = \frac{2A}{1-k^2}\dot{V}_1(t) + \dot{V}_2(t) \leq -\frac{1}{2}V_2(t), \quad \forall t \in [0,+\infty). \tag{3.43}$$

Notice that

$$\int_0^1 \xi^2(t,x)dx = \int_0^1 \rho^2(t,x)dx + (2d+d^2)W^2(t)$$

and

$$W^2(t) = \Big(\int_0^1 \rho(t,x)dx\Big)^2 \leq \int_0^1 \rho^2(t,x)dx.$$

Obviously, there exist constants $B_i = B_i(d) > 0$ $(i = 1,2,3,4)$ such that

$$B_1\|\rho(t,\cdot)\|^2_{L^2(0,1)} \leq B_2V_2(t) \leq V(t) \leq B_3V_2(t) \leq B_4\|\rho(t,\cdot)\|^2_{L^2(0,1)}, \quad \forall t \in [0,+\infty). \tag{3.44}$$

Now we conclude from (3.43) and (3.44) that there exist constants $\alpha = \alpha(\overline{\rho},k)$ and $C = C(\overline{\rho},k)$ such that

$$\dot{V}(t) \leq -\alpha V(t), \quad \forall t \in [0,+\infty) \tag{3.45}$$

and finally (3.4) holds. This finishes the proof of Theorem 3.1 for the case $d \geq 1$. $\square$

4 Stabilization to $\overline{\rho}$ for the nonlinear system

In this section, we stabilize the nonlinear system to an equilibrium $\overline{\rho} \in \mathbb{R}$ by using Lyapunov function approach. By Lemma 2.1 and Lemma 2.2, it suffices to construct Lyapunov functions for classical solutions. We will divide our main results into two cases: $\overline{\rho} = 0$ and $\overline{\rho} \neq 0$. We will see later that the situation of $\overline{\rho} \neq 0$ is much more complicated than that of $\overline{\rho} = 0$ which implies $d = 0$.

4.1 Exponential stability of 0 with a Lyapunov function approach

In this subsection, we prove a stabilization result for the case that $\overline{\rho} = 0$: we give explicit feedback laws leading to semi-global exponential stability of $\overline{\rho} = 0$.

Theorem 4.1. *For every $k \in (-1, 1)$ and every $R > 0$, there exist constants $C = C(k, R) > 0$ and $\alpha = \alpha(k, R) > 0$ such that for any $\rho_0 \in L^2(0, 1)$ with*

$$\|\rho_0\|_{L^1(0,1)} \leq R, \tag{4.1}$$

the solution $\rho \in C^0([0, +\infty); L^2(0, 1))$ to the Cauchy problem

$$\begin{cases} \rho_t(t, x) + (\rho(t, x)\lambda(W(t)))_x = 0, \quad t \in (0, +\infty), x \in (0, 1), \\ \rho(0, x) = \rho_0(x), \quad x \in (0, 1), \\ u(t) = ky(t), \quad t \in (0, +\infty) \end{cases} \tag{4.2}$$

satisfies

$$\|\rho(t, \cdot)\|_{L^2(0,1)} \leq Ce^{-\alpha t}\|\rho_0\|_{L^2(0,1)}, \quad \forall t \in [0, +\infty). \tag{4.3}$$

Proof. Still motivated by the Lyapunov functions used in [9, 13, 35, 37], we introduce the following Lyapunov function which is an weighted $L^2(0, 1)$ norm of the solution:

$$L(t) := \int_0^1 e^{-\beta x}\rho^2(t, x)dx, \quad \forall t \in [0, +\infty), \tag{4.4}$$

where $\beta > 0$ is a positive constant to be determined. Then, along the trajectories of (4.2) (see also (1.4) and (1.5)),

$$\begin{aligned} \dot{L}(t) =& \int_0^1 e^{-\beta x}(\rho^2(t, x))_t dx \\ =& -\lambda(W(t)) \int_0^1 e^{-\beta x}(\rho^2(t, x))_x dx \\ =& -\beta\lambda(W(t))L(t) - \lambda(W(t))\Big[e^{-\beta x}\rho^2(t, x)\Big]_{x=0}^{x=1} \\ =& -\beta\lambda(W(t))L(t) + \lambda(W(t))\big(\rho^2(t, 0) - e^{-\beta}\rho^2(t, 1)\big) \\ =& -\beta\lambda(W(t))L(t) + (\lambda(W(t)))^{-1}\big(k^2 - e^{-\beta}\big)y^2(t). \end{aligned}$$

Since $k \in (-1, 1)$, one can choose $\beta > 0$ so that $e^{-\beta} > k^2$ and thus

$$\dot{L}(t) = -\beta\lambda(W(t))L(t) \leq 0. \tag{4.5}$$

In order to get exponential decay of the solution as $t \to +\infty$, it suffices to prove the uniform boundedness of $W(\cdot)$. In fact,

$$t \mapsto \int_0^1 |\rho(t,x)|dx \text{ is a nonincreasing function.} \tag{4.6}$$

Property (4.6) can be proved by checking that, for every $r \in (1, 2]$,

$$\frac{d}{dt}\int_0^1 |\rho(t,x)|^r dx \leqslant 0, \tag{4.7}$$

From (4.7), one gets that, for every $r \in (1, 2]$,

$$t \mapsto \int_0^1 |\rho(t,x)|^r dx \text{ is a nonincreasing function,}$$

which gives (4.6) by letting $r \to 1^+$. From (4.6), one gets that

$$|W(t)| \leq \|\rho_0\|_{L^1(0,1)} \leq R, \quad \forall t \in [0, +\infty). \tag{4.8}$$

Let us define a constant

$$b := \inf_{|s| \leq R} \lambda(s) > 0. \tag{4.9}$$

Then we get from (4.1), (4.5), (4.8) and (4.9) that

$$\dot{L}(t) \leq -b\beta L(t), \quad \forall t \in [0, +\infty).$$

Therefore we obtain

$$L(t) \leq L(0)e^{-b\beta t} = \int_0^1 e^{-\beta x}\rho_0^2(x)dx\, e^{-b\beta t} \leq \|\rho_0\|^2_{L^2(0,1)}\, e^{-b\beta t}.$$

This concludes the proof of Theorem 4.1. □

Remark 4.1. It is easy to see that if we let $k = 0$, i.e., the feedback law is chosen as

$$u(t) = 0, \quad t \in (0, +\infty),$$

then the state reaches zero after a finite time no matter what the initial data is. This fact shows that zero control does drive the state to zero in finite time.

4.2 Exponential stability of $\overline{\rho} \neq 0$ with a Lyapunov function approach

In this subsection, our main result is the following one.

Theorem 4.2. *Assume that $d > -1$. Let $k \in (-1,1)$. Then there exist constants $\varepsilon = \varepsilon(\overline{\rho}, k) > 0$, $C = C(\overline{\rho}, k) > 0$ and $\alpha = \alpha(\overline{\rho}, k) > 0$ such that the following holds: For every $\rho_0 \in L^2(0,1)$ with*

$$\|\rho_0(\cdot) - \overline{\rho}\|_{L^2(0,1)} \leq \varepsilon, \tag{4.10}$$

the weak solution $\rho \in C^0([0,+\infty); L^2(0,1))$ to the Cauchy problem

$$\begin{cases} \rho_t(t,x) + (\rho(t,x)\lambda(W(t)))_x = 0, \quad t \in (0,+\infty), x \in (0,1), \\ \rho(0,x) = \rho_0(x), \quad x \in (0,1), \\ u(t) - \overline{\rho}\lambda(\overline{\rho}) = k(y(t) - \overline{\rho}\lambda(\overline{\rho})), \quad t \in (0,+\infty) \end{cases} \tag{4.11}$$

satisfies

$$\|\rho(t,\cdot) - \overline{\rho}\|_{L^2(0,1)} \leq Ce^{-\alpha t}\|\rho_0(\cdot) - \overline{\rho}\|_{L^2(0,1)}, \quad \forall t \in [0,+\infty). \tag{4.12}$$

Proof. As we did in Section 3.1 and Section 3.2, we may assume that $\lambda(\overline{\rho}) = 1$, which, together with (1.7), yields $d = \overline{\rho}\lambda'(\overline{\rho})$.

Let

$$\begin{aligned} &\widetilde{\rho}(t,x) := \rho(t,x) - \overline{\rho}, \quad \widetilde{W}(t) := W(t) - \overline{\rho}, \quad \widetilde{\rho}_0(x) := \rho_0(x) - \overline{\rho}, \\ &\widetilde{\lambda}(t) := \lambda(\overline{\rho} + \widetilde{W}(t)), \quad \widetilde{u}(t) := \widetilde{\lambda}(t)\widetilde{\rho}(t,0), \quad \widetilde{y}(t) := \widetilde{\lambda}(t)\widetilde{\rho}(t,1). \end{aligned}$$

The system (4.11) is then rewritten as follows

$$\begin{cases} \widetilde{\rho}_t(t,x) + \widetilde{\lambda}(t)\widetilde{\rho}_x(t,x) = 0, \quad t \in (0,+\infty), x \in (0,1), \\ \widetilde{\rho}(0,x) = \widetilde{\rho}_0(x), \quad x \in (0,1), \\ \widetilde{u}(t) = k\widetilde{y}(t) + (k-1)\overline{\rho}(\widetilde{\lambda}(t) - 1), \quad t \in (0,+\infty). \end{cases} \tag{4.13}$$

Until the end of the proof of theorem 4.2, we omit the symbol $\widetilde{\ }$. In particular we rewrite (4.13) as the following system

$$\begin{cases} \rho_t(t,x) + \lambda(t)\rho_x(t,x) = 0, \quad t \in (0,+\infty), x \in (0,1), \\ \rho(0,x) = \rho_0(x), \quad x \in (0,1), \\ u(t) = ky(t) + (k-1)\overline{\rho}(\lambda(t) - 1), \quad t \in (0,+\infty), \end{cases} \tag{4.14}$$

where

$$W(t) = \int_0^1 \rho(t,x)dx, \quad \lambda(t) = \lambda(\overline{\rho} + W(t)).$$

Correspondingly, the assumption (4.10) and the exponential decay estimate (4.12) become

$$\|\rho_0\|_{L^2(0,1)} \leq \varepsilon, \tag{4.15}$$

$$\|\rho(t,\cdot)\|_{L^2(0,1)} \leq Ce^{-\alpha t}\|\rho_0\|_{L^2(0,1)}, \quad \forall t \in [0,+\infty). \tag{4.16}$$

Similar to the linear case, we divide our proof into two cases: $|d| < 1$ and $d \geq 1$.

Case 1: $|d| < 1$.

We define a Lyapunov function by (3.22), where a is given by (3.28) and $\beta > 0$ is taken small enough so that (3.24) and (3.27) hold. Then, $L(t)$ is positive definite for every $t \geq 0$ and there exist four constants $C_i = C_i(d, k, \beta) > 0$ $(i = 1, 2, 3, 4)$ such that (3.25) holds.

Let us compute the time derivative of $L(t)$ for any classical solution of (4.14). Note that

$$\dot{W}(t) = \int_0^1 \rho_t(t,x)dx = \lambda(t)(\rho(t,0) - \rho(t,1)) = u(t) - y(t). \tag{4.17}$$

We get, from (3.22), (4.14) and (4.17), that

$$\begin{aligned}
\dot{L}(t) &= -\lambda(t) \int_0^1 e^{-\beta x} (\rho^2(t,x))_x dx + 2aW(t)\dot{W}(t) \\
&= -\beta\lambda(t) \int_0^1 e^{-\beta x} \rho^2(t,x)dx + \frac{u^2(t) - e^{-\beta} y^2(t)}{\lambda(t)} + 2aW(t)(u(t) - y(t)) \\
&= -\beta\lambda(t) \int_0^1 e^{-\beta x} \rho^2(t,x)dx + A_1,
\end{aligned} \tag{4.18}$$

where

$$\begin{aligned}
A_1 &= \frac{u^2(t) - e^{-\beta} y^2(t)}{\lambda(t)} + 2aW(t)(u(t) - y(t)) \\
&= \frac{[ky(t) + (k-1)\overline{\rho}(\lambda(t) - 1)]^2 - e^{-\beta} y^2(t)}{\lambda(t)} + 2a(k-1)W(t)[y(t) + \overline{\rho}(\lambda(t) - 1)] \\
&= \frac{k^2 - e^{-\beta}}{\lambda(t)} \cdot \left[y(t) + \frac{(k-1)[k\overline{\rho}(\lambda(t) - 1) + a\lambda(t)]W(t)}{k^2 - e^{-\beta}} \right]^2 - \frac{(k-1)^2 a^2 \lambda^2(t) W^2(t)}{\lambda(t)(k^2 - e^{-\beta})} \\
&\quad - \frac{2a(e^{-\beta} - k)(k-1)\lambda(t)W(t)\overline{\rho}(\lambda(t) - 1) + e^{-\beta}(k-1)^2 \overline{\rho}^2 (\lambda(t) - 1)^2}{\lambda(t)(k^2 - e^{-\beta})}.
\end{aligned} \tag{4.19}$$

Since λ is of class C^1, one has

$$\lambda(t) = \lambda(\overline{\rho} + W(t)) = 1 + \lambda'(\overline{\rho})W(t) + o(1)W(t), \quad \forall t \in [0, +\infty). \tag{4.20}$$

Here and hereafter, we denote $o(1)$ for various quantities (may be different in different situations) satisfying the following property

$$\forall \delta > 0, \exists \varepsilon > 0 \text{ such that } \Big((|W(t)| \leq \varepsilon \Rightarrow |o(1)| \leq \delta), \quad \forall t \in [0, +\infty) \Big). \tag{4.21}$$

Then, using (3.28), (4.19) and (4.20), one has

$$\begin{aligned}
A_1 &\leq \frac{[(k-1)^2 a^2 + 2a(e^{-\beta} - k)(k-1)d + e^{-\beta}(k-1)^2 d^2] W^2(t) + o(1) W^2(t)}{(e^{-\beta} - k^2)(1 + o(1))} \\
&\leq [d^2(1 - e^{-\beta}) + o(1)] W^2(t).
\end{aligned} \tag{4.22}$$

Combining (3.23), (3.25), (4.18), (4.20), (4.21) and (4.22), we get

$$\begin{aligned}
\dot{L}(t) &\leq \Big(-\beta\lambda(t) + d^2(e^\beta - 1)^2 e^{-\beta} \beta^{-1} + o(1) \Big) \int_0^1 e^{-\beta x} \rho^2(t,x)dx \\
&\leq -\frac{\beta}{C_3} \Big(1 - d^2(e^\beta - 1)^2 e^{-\beta} \beta^{-2} + o(1) \Big) L(t).
\end{aligned} \tag{4.23}$$

From the fact that $1 - d^2(e^\beta - 1)^2 e^{-\beta}\beta^{-2} \to 1 - d^2 > 0$ as $\beta \to 0^+$, we have, for $\beta > 0$ small enough,

$$1 - d^2(e^\beta - 1)^2 e^{-\beta}\beta^{-2} > 0. \tag{4.24}$$

Finally, we take $\beta = \beta(\overline{\rho}, k) > 0$ small enough so that (3.24) and (4.24) hold. Let us also emphasize that, from the Cauchy-Schwarz inequality and (3.25), there exits $C = C(\overline{\rho}, k) > 0$ such that

$$|W(t)|^2 \leq CL(t), \tag{4.25}$$

which, together with (4.15), (4.21), (4.23) and (4.24) concludes the proof of Theorem 4.2 in the case $|d| < 1$.

Case 2: $d \geq 1$.

In this case, we are going to prove that the Lyapunov function in the type of (3.42) still works for the nonlinear control system (4.14).

Let's first compute the time derivative of $V_1(t)$ (see (3.33) for definition), using (3.35), (4.14), (4.17) and (4.20):

$$\begin{aligned}
\dot{V}_1(t) &= -\lambda(t)\int_0^1 (\rho^2(t,x))_x dx + 2dW(t)\dot{W}(t) \\
&= \lambda(t)[\rho^2(t,0) - \rho^2(t,1) + 2dW(t)(\rho(t,0) - \rho(t,1))] \\
&= \frac{[ky(t) + (k-1)\overline{\rho}(\lambda(t)-1)]^2 - y^2(t)}{\lambda(t)} + 2d(k-1)W(t)(y(t) + \overline{\rho}(\lambda(t)-1)) \\
&= \lambda(t)(k^2-1)(\rho(t,1) + dW(t))^2 + [o(1)W(t)]\rho(t,1) + o(1)W^2(t) \\
&= \lambda(t)(k^2-1)\xi^2(t,1) + o(1)W(t)\xi(t,1) + o(1)W^2(t).
\end{aligned}$$

Thanks to the Cauchy-Schwarz inequality and (3.44), it follows that

$$\dot{V}_1 \leq (k^2 - 1 + o(1))\xi^2(t,1) + o(1)W^2(t). \tag{4.26}$$

By the definition of ξ, it is easy to get that ξ satisfies the following Cauchy problem

$$\begin{cases}
\xi_t(t,x) + \lambda(t)\xi_x(t,x) = d\dot{W}(t), \quad t \in (0,+\infty), x \in (0,1), \\
\xi(0,x) = \rho_0(x) + dW(0), \quad x \in (0,1), \\
\xi(t,0) = k\xi(t,1) + (k-1)\left(\dfrac{\overline{\rho}(\lambda(t)-1)}{\lambda(t)} - dW(t)\right), \quad t \in (0,+\infty).
\end{cases} \tag{4.27}$$

Let, again, V_2 be defined by (3.38). Then the time derivative of V_2 satisfies:

$$\begin{aligned}
\dot{V}_2(t) &= -\lambda(t)\int_0^1 e^{-x}(\xi^2(t,x))_x dx + 2d\dot{W}(t)\int_0^1 e^{-x}\xi(t,x)dx \\
&= -\lambda(t)\int_0^1 e^{-x}\xi^2(t,x)dx - \lambda(t)\Big[e^{-x}\xi^2(t,x)\Big]_{x=0}^{x=1} \\
&\quad + 2d\lambda(t)(\xi(t,0) - \xi(t,1))\int_0^1 e^{-x}\xi(t,x)dx.
\end{aligned}$$

Hence

$$\dot{V}_2(t) = -\lambda(t)V_2(t) - \lambda(t)\left[e^{-1}\xi^2(t,1) - \left(k\xi(t,1) + (k-1)\Big(\frac{\overline{\rho}(\lambda(t)-1)}{\lambda(t)} - dW(t)\Big)\right)^2\right]$$
$$+ 2d(k-1)\lambda(t)\Big(\xi(t,1) + \frac{\overline{\rho}(\lambda(t)-1)}{\lambda(t)} - dW(t)\Big)\int_0^1 e^{-x}\xi(t,x)dx$$
$$= -\lambda(t)V_2(t) + \lambda(t)(k^2 - e^{-1})\xi^2(t,1) + o(1)W(t)\xi(t,1) + o(1)W^2(t)$$
$$+ 2d(k-1)\lambda(t)(\xi(t,1) + o(1)W(t))\int_0^1 e^{-x}\xi(t,x)dx. \tag{4.28}$$

By the Cauchy-Schwarz inequality, (3.40), (3.44) and (4.28), one has, for $A = A(\overline{\rho},k) > 0$ sufficiently large,

$$\dot{V}_2(t) \leq (-1 + o(1))V_2(t) + A(1 + o(1))\xi^2(t,1) + o(1)W^2(t). \tag{4.29}$$

We still define V by (3.42). Combining (4.26) and (4.29), one has

$$\begin{aligned}\dot{V}(t) &= \frac{2A}{1-k^2}\dot{V}_1(t) + \dot{V}_2(t)\\ &\leq (-1 + o(1))V_2(t) + (-A + o(1))\xi^2(t,1) + o(1)W^2(t).\end{aligned} \tag{4.30}$$

Consequently, there exists $\alpha = \alpha(\overline{\rho},k) > 0$ such that

$$\dot{V}(t) \leq (-\alpha + o(1))V(t). \tag{4.31}$$

This, together with (3.42) (3.44), (4.15) and (4.21), concludes the proof of Theorem 4.2 in the case $d \geq 1$. □

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